Improvement of the spherical harmonics method convergence at strongly anisotropic scattering

V.P. Boudak, A.V. Kozelsky, and E.N. Savitsky

Moscow Energetic Institute (Technical University)

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The algorithm for convergence acceleration of the spherical harmonics method based on the small angle approximation (SAA) considering the part of solution, anisotropic over the sighting angle, is offered. On the basis of the numerical solution of the radiative transfer equation (RTE) by the method of spherical harmonics, the difference between the RTE and SAA precise solutions is found. The use of the small-angle modification of the spherical harmonics method as the SAA allows us to obtain the solution in the analytic form that demonstrates high potentiality of the algorithm.

Introduction

Natural formations like the atmosphere and sea water contain suspended particles with the sizes essentially greater than the light wavelength, which causes strongly anisotropic light scattering. Inconvenience of application of big computers and complicated algorithms to the engineering practice stimulated the development of special approximation methods for solving boundary problems of the radiative transfer equation (RTE) generally called the small-angle approximation (SAA) methods.[1] However, the rapid recent development of computers and numerical methods make one to reappraise the role of approximate methods in the radiative transfer theory. Today, solving the boundary problem of RTE by numerical methods often requires no more time than the small-angle approximation calculation of the light field under the same conditions.

Nevertheless, difficulties in solving the RTE boundary problems under conditions of strong anisotropy still persist, and they supposedly are of a fundamental nature connected with mathematical incorrectness in solving the problem in this case. The poorly conditioned matrix in the method of spherical harmonics (MSH) leads to strong oscillations in the solution; in the Monte-Carlo method, backscattering is a low-probable event with high weight. In the SAA, the backscattering is found by the method of disturbances, that makes the accuracy of calculations equivalent to the accuracy of the transport approximation.

In this paper we propose an approach, in which the difference between the accurate solution of RTE and SAA is determined on the basis of numerical solution of RTE. Since the SAA contains all peculiarities of the accurate solution, the difference is a smooth function, numerical determination of which should not present any difficulties. The type and analytical form of the SAA determine the numerical method to be used. We take the small-angle modification of the MSH as the initial solution.[3] Analytically, it is represented by a series of spherical harmonics that determines the MSH as the numerical method.[4]

The boundary problem of the RTE for a layer of turbid liquid of the optical thickness τ_0 illuminated by a plane monodirected source (PMS) in the direction $\hat{\mathbf{l}}_0 = \{\sqrt{1-\mu_0^2}, 0, \mu_0\}$, $\mu_0 = \cos\vartheta_0$, has the form

$$\begin{cases} \mu \dfrac{dL(\tau,\mu,\varphi)}{d\tau} + L(\tau,\mu,\varphi) - \dfrac{\Lambda}{4\pi} \oint x(\hat{\mathbf{l}},\hat{\mathbf{l}}') L(\tau,\mu',\varphi') d\hat{\mathbf{l}}' = 0, \\ L(\tau,\mu,\varphi)\big|_{\tau=0,\,\mu\ge 0} = \delta(\hat{\mathbf{l}} - \hat{\mathbf{l}}_0), \; L(\tau,\mu,\varphi)\big|_{\tau=\tau_0,\,\mu\le 0} = 0, \end{cases} \quad (1)$$

where $L(\tau,\mu,\varphi) \equiv L(\tau, \hat{\mathbf{l}})$ is the brightness of the light field at the optical depth τ in the direction of vision $\hat{\mathbf{l}} = \{\sqrt{1-\mu^2}\cos\varphi, \sqrt{1-\mu^2}\sin\varphi, \mu\}$, $\mu = \cos\vartheta$; the axis OZ is perpendicular to the layer boundary, Λ is the single scattering albedo, $x(\hat{\mathbf{l}}, \hat{\mathbf{l}}')$ is the scattering phase function. The sign "corner" above a letter here and further denotes a unit vector.

The essence of the SH method is the expansion of the sought brightness of the light field and the scattering phase function in spherical functions, resulting in the infinite system of differential equations with constant coefficients.[3] To solve the system, we transform it into finite one through setting all coefficient of the field expansion equal to zero starting from the numbers greater some N (P_N approximation). However, in the case of illumination by a PMS, the solution involves a singularity determined by the direct non-scattered radiation of the source. Therefore, at any N the solution is strongly smoothed, that leads to its significant oscillations. To remove this, the direct radiation is subtracted from the solution, and the boundary problem is formulated only for the scattered radiation.[4] It is easily seen that the expansion of the scattered radiation has the same order of expansion as the scattering phase function.

The system of differential equations with constant coefficients in the MSH has an analytical solution in the form of a linear combination of exponents with the indices, which are the matrix eigenvalues of the system.[3] As the matrix has paired eigenvalues with different signs and strongly different absolute values, the solution of the system becomes unstable already at $N > 30$ (Ref. 5). To make the solution stable, the step-by-step orthogonalization of the solution was proposed (Ref. 6), which then was elegantly finalized in the analytical form of the similarity transformation.[3] The method of spherical harmonics in this form allows the solution at any N; but at the incidence angles different from normal, there appear oscillations due to limitation of the quantity of azimuth harmonics. This has required different smoothing procedures,[7,8] which turned to be

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time-consuming and introducing uncontrolled arbitrariness into the rigorous algorithm of the numerical solution.

To remove the noted instability, let us present the boundary problem (1) in the form of the sum

$$L(\tau,\mu,\varphi) = \tilde{L}(\tau,\mu,\varphi) + L_{\text{SAA}}(\tau,\hat{\mathbf{l}},\hat{\mathbf{l}}_0), \tag{2}$$

the solution of which in the small-angle approximation[9] has the form

$$L_{\text{SAA}}(\tau,\hat{\mathbf{l}},\hat{\mathbf{l}}_0) = \sum_{k=0}^{\infty}\frac{2k+1}{4\pi}Z_k(\tau)P_k(\hat{\mathbf{l}}\hat{\mathbf{l}}_0) =$$

$$= \sum_{k=0}^{\infty}\frac{2k+1}{4\pi}Z_k(\tau)\sum_{m=0}^{k}(2-\delta_{m0})Q_k^m(\mu_0)Q_k^m(\mu)\cos m\varphi, \tag{3}$$

where

$$Z_k(\tau) = \exp\left\{-\frac{(1-\Lambda x_k)\tau}{\mu_0}\right\};$$

$P_l^n(\mu)$ are the adjoined Legendre polynomials,

$$P_l(\mu) \equiv P_l^0(\mu);$$

$$Q_l^n(\mu) = \sqrt{\frac{(l-n)!}{(l+n)!}}P_l^n(\mu)$$

are the renormalized Legendre polynomials,[3] for which the following normalization is valid

$$\int_{-1}^{1}Q_l^n(\mu)Q_k^n(\mu)\mathrm{d}\mu = \frac{2}{2k+1}\delta_{lk}. \tag{4}$$

The solution (3) meets the boundary problem similar to (1) but with boundary conditions neglecting the backscattering

$$L_{\text{SAA}}(\tau,\hat{\mathbf{l}},\hat{\mathbf{l}}_0)\Big|_{\tau=0} = \delta(\hat{\mathbf{l}}-\hat{\mathbf{l}}_0) \tag{5}$$

contains the whole anisotropic part of the field, that makes $\tilde{L}(\tau,\mu,\varphi)$ a smooth function meeting the boundary problem:

$$\begin{cases} \mu\dfrac{\mathrm{d}\tilde{L}(\tau,\mu,\varphi)}{\mathrm{d}\tau} + \tilde{L}(\tau,\mu,\varphi) - \dfrac{\Lambda}{4\pi}\oint x(\hat{\mathbf{l}},\hat{\mathbf{l}}')\tilde{L}(\tau,\mu',\varphi')\mathrm{d}\hat{\mathbf{l}}' = \\ = -\mu\dfrac{\mathrm{d}L_{\text{SAA}}(\tau,\hat{\mathbf{l}},\hat{\mathbf{l}}_0)}{\mathrm{d}\tau} - L_{\text{SAA}}(\tau,\hat{\mathbf{l}},\hat{\mathbf{l}}_0) + \\ +\dfrac{\Lambda}{4\pi}\oint x(\hat{\mathbf{l}},\hat{\mathbf{l}}')L_{\text{SAA}}(\tau,\hat{\mathbf{l}},\hat{\mathbf{l}}_0)\mathrm{d}\hat{\mathbf{l}}'; \\ L(\tau,\mu,\varphi)\big|_{\tau=0,\,\mu\ge 0} = 0, \quad L(\tau,\mu,\varphi)\big|_{\tau=\tau_0,\,\mu\le 0} = -L_{\text{SAA}}(\tau_0,\hat{\mathbf{l}},\hat{\mathbf{l}}_0). \end{cases} \tag{6}$$

Represent the solution and the scattering phase function in the form of expansion in terms of spherical harmonics

$$\begin{gathered} \tilde{L}(\tau,\mu,\varphi) = \sum_{l=0}^{\infty}\sum_{n=0}^{l}\frac{2l+1}{4\pi}C_l^n(\tau)Q_l^n(\mu)e^{-in\varphi}, \\ x(\hat{\mathbf{l}},\hat{\mathbf{l}}') = \sum_{l=0}^{\infty}(2l+1)x_lP_l(\hat{\mathbf{l}},\hat{\mathbf{l}}'). \end{gathered} \tag{7}$$

Due to the light field symmetry relative to the plane OZX the field expansion coefficients in Eq. (7) should have the symmetry $C_l^n(\tau) = C_l^{-n}(\tau)$, that leads Eq. (7) to the classical expansion[4]:

$$L(\tau,\mu,\varphi) = \sum_{n=0}^{\infty}(2-\delta_{0n})\cos n\varphi\sum_{l=n}^{\infty}\frac{2l+1}{4\pi}C_l^n(\tau)Q_l^n(\mu).$$

Taking into account the addition theorem for Legendre polynomials, the scattering phase function in Eq. (7) can be rewritten in the form

$$x(\hat{\mathbf{l}},\hat{\mathbf{l}}') = 4\pi\sum_{l=0}^{\infty}\sum_{n=-l}^{l}x_lY_l^n(\hat{\mathbf{l}})\overline{Y_l^n(\hat{\mathbf{l}}')}, \tag{8}$$

where $Y_l^n(\hat{\mathbf{l}}') = \sqrt{\dfrac{2k+1}{2}}Q_k^m(\mu)e^{-im\varphi}$ are the spherical functions forming a full orthonormalized system for the sphere, the line above denotes the complex conjugation.

Substitute the formulas (3) and (7) into RTE of the boundary problem (6) and take into account Eq. (8). Then we multiply the obtained equation by $Q_k^m(\mu)\,e^{-im\varphi}$ and integrate over all range of μ and φ variation. Then, accounting for the orthogonality of spherical functions, we obtain the infinite system of linked differential equations of the MSH

$$\begin{gathered} \frac{1}{2k+1}\frac{\mathrm{d}}{\mathrm{d}\tau}\Big[\sqrt{(k-m+1)(k+m+1)}\,C_{k+1}^m + \\ +\sqrt{(k-m)(k+m)}C_{k-1}^m\Big] + (1-\Lambda x_k)C_k^m(\tau) = \\ = -\frac{1}{2k+1}\frac{1}{\mu_0}\frac{\mathrm{d}}{\mathrm{d}\tau}\Big[\sqrt{(k-m+1)(k+m+1)}\times \\ \times(1-\Lambda x_{k+1})Q_{k+1}^m(\mu_0)Z_{k+1} + \\ +\sqrt{(k-m)(k+m)}(1-\Lambda x_{k-1})Q_{k-1}^m(\mu_0)Z_{k-1}\Big] - \\ -(1-\Lambda x_k)Q_k^m(\mu_0)Z_k(\tau). \end{gathered} \tag{9}$$

Solution of the infinite system (9) is impossible, therefore, it should be transformed into finite through setting $C_k^m(\tau) \equiv 0,\ \forall k \ge N$. However, when solving the boundary problem, the well-known problem appears: the approximate solution fails to satisfy the accurate boundary conditions, and they should be replaced with approximate ones. The best variant is the boundary conditions in the Marshak form[10] representing the energy conservation law in the form of equality of the radiation fluxes at the boundary of the medium:

$$\forall j \in \overline{\frac{m+1}{2}, \frac{N+1}{2}},\ \forall m \in \overline{0, N}:$$

$$\int\limits_{\Omega_+} L(0,\hat{\mathbf{l}})Y_{2j-1}^m(\hat{\mathbf{l}})\mathrm{d}\hat{\mathbf{l}} = 0,$$

$$\int\limits_{\Omega_-} L(\tau_0,\hat{\mathbf{l}})Y_{2j-1}^m(\hat{\mathbf{l}})\mathrm{d}\hat{\mathbf{l}} = -\int\limits_{\Omega_-} L_{\mathrm{SAA}}(\tau_0,\hat{\mathbf{l}},\hat{\mathbf{l}}_0)Y_{2j-1}^m(\hat{\mathbf{l}})\mathrm{d}\hat{\mathbf{l}}, \qquad (10)$$

where $\mathbf{\Omega}_\pm = \{\hat{\mathbf{l}} : (\hat{\mathbf{l}},\hat{\mathbf{z}}) \gtrless 0\}$, $\hat{\mathbf{z}}$ is the unit vector normal to the boundary.

The finite system of differential equations (9) with constant coefficients has $N + 1 - m$ constants in the solution, which are determined from the system of boundary conditions (10). In practice, odd N are better for even m and *vice versa*.[3] So further it will be assumed that the system (9) has $N_m = N + \delta_m$ equations, where

$$\delta_m = \begin{cases} 1, \text{ for even } m, \\ 0, \text{ for odd } m. \end{cases}$$

Taking into account Eqs. (3) and (7), the boundary conditions (10) have the form

$$C_{2j+1}^m(0) + \sum_{l=m/2-1}^{\frac{N+\delta_m}{2}-1} G_{jl}^m C_{2l-2}^m(0) = 0,$$

$$C_{2j+1}^m(\tau_0) - \sum_{l=m/2-1}^{\frac{N+\delta_m}{2}-1} G_{jl}^m C_{2l-2}^m(\tau_0) = -Y_j^m(\tau_0), \qquad (11)$$

where

$$Y_j^m(\tau,\mu_0) = \sum_{k=0}^{\infty}(2k+1)Z_k(\tau)Q_k^m(\mu_0)\int\limits_{-1}^{0} Q_k^m(\mu)Q_{2j-1}^m(\mu)\mathrm{d}\mu =$$

$$= Z_{2j-1}(\tau)Q_{2j-1}^m(\mu_0) - \sum_{l=0}^{\infty} G_{jl}^m Z_{2l-2}(\tau)Q_{2l-2}^m(\mu_0),$$

for coefficients

$$G_{jl} = (4l-3)\int\limits_0^1 Q_{2j-1}^m(\mu)Q_{2l-2}^m(\mu)\mathrm{d}\mu$$

simple recurrent relationships are presented in Ref. 11.

For analytical convenience, we present the finite system (9) in a matrix form as in Ref. 7:

$$\overleftrightarrow{A}^m \frac{\mathrm{d}}{\mathrm{d}\tau}\vec{C}^m(\tau) + \overleftrightarrow{D}\,\vec{C}^m(\tau) =$$

$$= (\overleftrightarrow{A}^m/\mu_0 - \overleftrightarrow{1})\overleftrightarrow{D}\,\overleftrightarrow{Q}^m\vec{Z}(\tau) + a_{N+1}^m\vec{Z}_{N+1}(\tau), \qquad (12)$$

where

$$\left(\overleftrightarrow{A}^m\right)_{i,i+1} = \frac{\sqrt{(i-m)(i+m)}}{2i-1},$$

$$\left(\overleftrightarrow{A}^m\right)_{i,i-1} = \frac{\sqrt{(i-m-1)(i+m-1)}}{2i-1};$$

$$\vec{C} = \left\{C_{i-1}^m(\tau)\right\},\quad \overleftrightarrow{D} = \mathrm{Diag}\left\{(1-\Lambda x_{i-1})/\mu_0\right\};$$

$$\overleftrightarrow{Q}^m = \mathrm{Diag}\left\{Q_{i-1}^m(\mu_0)\right\};\quad \vec{Z} = \left\{Z_{i-1}(\tau)\right\};$$

$$a_{N+1}^m = \frac{\sqrt{(N+1-m)(N+1+m)}}{2N+1}\,\frac{1-\Lambda x_{N+1}}{\mu_0}Q_{N+1}^m(\mu_0);$$

$$\vec{Z}_{N+1} = \left\{\underbrace{0\ldots0}_{N}, Z_{N+1}(\tau)\right\}.$$

The double arrow above a symbol here and below denotes a matrix, single right arrow denotes a vector-column, and single left arrow denotes a vector-row. For simplicity, we omit the azimuth index m, where it is obvious.

In P_N approximation of the MSH, the coefficient $Z_{N+1}(\tau)$, accounting for Eq. (9), enters to the right part of the equation. The matrix $\overleftrightarrow{A}^m$ does not directly takes it into account, because it is formed on the assumption that $C_{N+1}^m(\tau) = 0$. This determines the last term in the right part of the equation (12).

The solution of the system (12) can be represented in the form

$$-\vec{C}(0) + e^{-\overleftrightarrow{B}\tau_0}\vec{C}(\tau_0) = \frac{1}{\mu_0}\int\limits_0^{\tau_0} e^{-\overleftrightarrow{B}t}(\overleftrightarrow{1} - \mu_0\overleftrightarrow{A}^{-1})\overleftrightarrow{D}\,\overleftrightarrow{Q}\vec{Z}(t)\mathrm{d}t +$$

$$+ a_{N+1}^m\int\limits_0^{\tau_0} e^{-\overleftrightarrow{B}t}\,\overleftrightarrow{A}^{-1}\vec{Z}_{N+1}(t)\mathrm{d}t, \qquad (13)$$

where $\overleftrightarrow{B} = \overleftrightarrow{A}^{-1}\overleftrightarrow{D}$.

The matrix exponent always can be represented in the form

$$e^{-\overleftrightarrow{B}t} = \overleftrightarrow{U}\,e^{-\overleftrightarrow{\Gamma}t}\,\overleftrightarrow{U}^{-1}, \qquad (14)$$

where $\overleftrightarrow{\Gamma}$ is the diagonal matrix of eigenvalues, $\overleftrightarrow{U}$ is the matrix of corresponding eigenvectors of the matrix $\overleftrightarrow{B}$.

Taking into account the last formula (14), the integral in Eq. (13) can be rewritten as:

$$\overleftrightarrow{J}(\tau) = \frac{1}{\mu_0}\int\limits_0^{\tau_0} e^{-\overleftrightarrow{\Gamma}t}\,\overleftrightarrow{T}\vec{Z}(t)\mathrm{d}t +$$

$$+ a_{N+1}^m\int\limits_0^{\tau_0} e^{-\overleftrightarrow{\Gamma}t} Z_{N+1}(t)\mathrm{d}t\,\overleftrightarrow{U}^{-1}\overleftrightarrow{A}_{N+1}^{-1}, \qquad (15)$$

where $\overleftrightarrow{T} = \overleftrightarrow{U}^{-1}(\overleftrightarrow{1} - \mu_0\overleftrightarrow{A}^{-1})\overleftrightarrow{D}\,\overleftrightarrow{Q}$.

Since $e^{-\overleftrightarrow{\Gamma}t}$ is the diagonal matrix, and

$$\vec{Z}(t) = \left\{\exp\left[-(1-\Lambda x_{i-1})/\mu_0\right]\right\}$$

is the vector-column, all elements of the matrix $\overleftrightarrow{J}$ can be readily calculated

$$\vec{J} = \left\{ \sum_{j=1}^{N+1} \left(\hat{T}\right)_{ij} \frac{\left[1 - \exp\left[-\Gamma_{ii}\tau_0 - (1 - \Lambda x_{j-1})\tau_0 / \mu_0\right]\right]}{\Gamma_{ii}\mu_0 + (1 - \Lambda x_{j-1})} \right\} +$$

$$+ a_{N+1}^{m}\left(\hat{\Gamma} + b_{N+1}\hat{1}\right)^{-1}\left\{\hat{1} - \exp\left[-(\hat{\Gamma} + b_{N+1}\hat{1})\right]\right\}\hat{U}^{-1}\vec{A}_{N+1}^{-1}, \tag{16}$$

and the formula (13) takes the form

$$-\vec{C}(0) + \hat{U}\, e^{-\hat{\Gamma} t}\, \hat{U}^{-1}\vec{C}(\tau_0) = \hat{U}\vec{J}(\tau_0) . \tag{17}$$

The formula (17) is the system of $(N_m - m)$ linear algebraic equations from $2(N_m - m)$. The deficient $(N_m - m)$ equations are provided by the boundary conditions (11). We give them a matrix form, analogously to Ref. 6:

$$\vec{C}_{\mathrm{odd}}(0) + \hat{G}\vec{C}_{\mathrm{even}}(0) = 0,$$

$$\vec{C}_{\mathrm{odd}}(\tau_0) - \hat{G}\vec{C}_{\mathrm{even}}(\tau_0) = -\vec{Y}(\tau_0, \mu_0), \tag{18}$$

where

$$\left(\hat{G}\right)_{jl} = G_{jl};\ \vec{Y}(\tau_0, \mu_0) = \left\{Y_j^m(\tau_0, \mu_0)\right\},$$

and the indices "odd" and "even" denote the columns composed from odd and even elements of $\vec{C}$, respectively.

It is possible to eliminate the problem of separation of odd and even elements in the boundary conditions through introducing the matrix, which sorts the columns to even and odd elements[6]:

$$\hat{P}\vec{C} \equiv \begin{bmatrix} \vec{C}_{\mathrm{odd}} \\ \vec{C}_{\mathrm{even}} \end{bmatrix},$$

that allows the boundary conditions to have the form

$$\begin{bmatrix} \hat{1} & \hat{G} \end{bmatrix}\hat{P}\vec{C}(0) = \vec{0},\ \begin{bmatrix} \hat{1} - \hat{G} \end{bmatrix}\hat{P}\vec{C}(\tau_0) = \begin{bmatrix} \vec{0} \\ -\vec{Y}(\tau_0) \end{bmatrix}. \tag{19}$$

Then the formulae (17) and (19) form a closed system of linear algebraic equations, from solution of which we can determine the sought coefficients. The matrix conditionality of the system quickly deteriorates as the layer thickness increases. To remove this, we use the scale transformation.[3] Assume that the eigenvalues of $\hat{B}$ are sorted in ascending order

$$\hat{\Gamma} = \begin{bmatrix} \hat{\Gamma}_- & 0 \\ 0 & \hat{\Gamma}_+ \end{bmatrix},\ \hat{\Gamma}_+ = \hat{\Gamma}_- = \mathrm{Diag}\{\gamma_i\},\ \gamma_i < \gamma_{i+1}.$$

Then

$$e^{-\hat{\Gamma}\tau} = \begin{bmatrix} e^{-\hat{\Gamma}_-\tau} & 0 \\ 0 & e^{-\hat{\Gamma}_+\tau} \end{bmatrix}. \tag{20}$$

For similarity transformation we multiply Eq. (17) by $\hat{S}\hat{U}^{-1}$, where

$$\hat{S} = \begin{bmatrix} 0 & \hat{1} \\ e^{\hat{\Gamma}_-\tau_0} & 0 \end{bmatrix}. \tag{21}$$

Then, taking into account Eqs. (20) and (21), Eq. (17) takes the form

$$-\hat{S}\hat{U}^{-1}\vec{C}(0) + \hat{H}\,\hat{U}^{-1}\vec{C}(\tau_0) = \hat{S}\vec{J}(\tau_0) , \tag{22}$$

where

$$\hat{H} = \hat{S}e^{-\hat{\Gamma}\tau_0} = \begin{bmatrix} 0 & \hat{1} \\ e^{\hat{\Gamma}_-\tau_0} & 0 \end{bmatrix}\begin{bmatrix} e^{-\hat{\Gamma}_-\tau_0} & 0 \\ 0 & e^{-\hat{\Gamma}_+\tau_0} \end{bmatrix} = \begin{bmatrix} 0 & e^{-\hat{\Gamma}_+\tau_0} \\ \hat{1} & 0 \end{bmatrix}.$$

It is easily seen that the exponents with only negative indices enter the equation (22), and the equation keeps stability at any N and τ_0. The sought coefficients of expansion for the brightness body at the upper and lower boundaries of the medium can be determined from the solution of the system of linear equations (19) and (22).

The proposed algorithm holds at $\Lambda \neq 1$. At $\Lambda = 1$ $\hat{B}$ degenerates, and the form (14) is impossible. However, the formula (14) can be replaced with the Jordan form. This, in fact, changes nothing in the algorithm, but complicates its representation. At $\Lambda = 1$, the algorithm is constructed in the way similar to those described in Ref. 3 with accounting for the change of the right part of Eq. (22) in accordance with Eq. (16). The method of solution can be readily generalized to the case of the stratified layer: the layer is divided into a set of homogeneous layers, and a general system similar to the foregoing is written for each of them. The algorithm was realized in the MathWorks Matlab v.6.5 Release 13 system, which offers a simple interface to any matrix operations. The calculation time at the Intel Pentium IV 2.4 GHz computer does not exceed 30 s at any input data.

As the accurate solution for leading hemisphere at small optical thickness differs little from the small-angle approximation,[9] formula (3) allows us to estimate the quantity of the necessary azimuth harmonics. In Eq. (3), it is easily seen that the coefficient of expansion of the RTE solution for PMS is roughly equal to $(2 - \delta_{m0})Z_k(\tau)\,Q_k^m(\mu_0)$, i.e., the m dependence is determined by $Q_k^m(\mu_0)$, that, in general case, at great incidence angles requires that m . k, and k is determined by the degree of anisotropy of the scattering phase function.

To determine the relation between optical and microphysical characteristics of the medium (particle size distribution and refractive index of particles), the program was complemented with the results of the Mie theory,[12] i.e., the expansion of the scattering phase function directly in terms of spherical functions.

The calculation results for the light fields are shown in Figs. 1–4. They clearly demonstrate the acceleration of the spherical harmonics method convergence when using our algorithm.

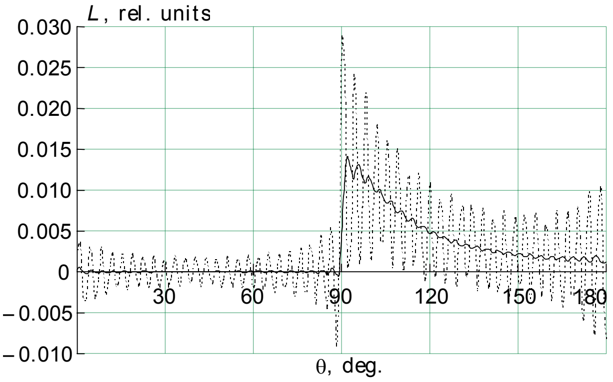

Fig. 1. Brightness of radiation reflected from the layer at $\tau_0 = 20.0$, $\Lambda = 0.8$, $\theta_0 = 40°$. Quantity of the Legendre polynomials in the expansion $N = 99$, $M = 4$. Dashed lines show the calculations by MSH.[3]

Solid line in all plots shows the results of calculation by our algorithm. For simplicity of the analysis, all calculations were performed for the Henyey-Greenstein scattering phase function depending on the only parameter g, the mean cosine of the scattering angle. All plots correspond to $g = 0.97$, which is a good representation for ocean and cloud scattering phase functions.

Figures 1 and 2 demonstrate the acceleration of convergence when calculating the radiation reflected from a layer: twice more harmonics are required for reaching approximately the same calculation accuracy with MSH,[3] that increases the calculation time by more then 4 times. Both algorithms have an equal calculation accuracy at $N = 299$ and $M = 8$.

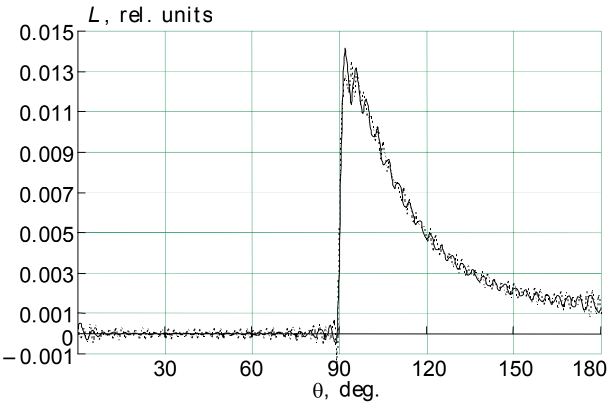

Fig. 2. The same as in Fig. 1 at calculations by MSG (Ref. 3): $N = 199$, $M = 8$.

Figure 3 shows the comparison of brightness of the reflected radiation at these values of parameters with calculations in the single scattering and "quasi-single scattering" approximations. In the latter case, the backscattering is taken into account in the single-scattering approximation, and forward scattering – in the small-angle approximation. It is easy to see that the quasi-single scattering approximation significantly overestimates the brightness value.

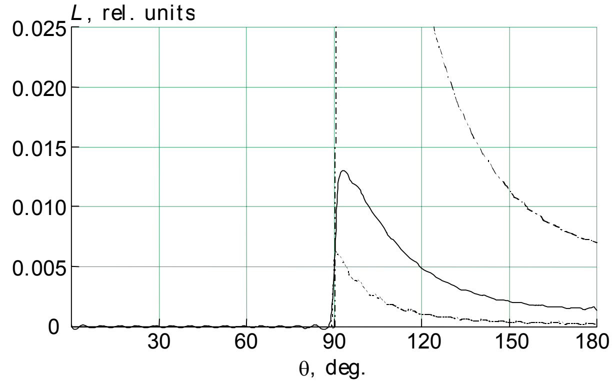

Fig. 3. Brightness of radiation reflected from the layer at $\tau_0 = 20.0$, $\Lambda = 0.8$, $\theta_0 = 40°$. Quantity of the Legendre polynomials in the expansion $N = 299$, $M = 8$. Dashed lines show the calculations in the single-scattering approximation, dash-dot line shows the small-angle approximation.

Comparison of calculations for passed radiation is shown in Fig. 4. The calculations were carried out at $\tau = 0.5$, when the single scattering approximation involved into the algorithm proposed by us can be assumed coinciding with the accurate RTE solution. It is easily seen that the MSH[3] meets the solution by our algorithm with the increase of the number of harmonics, however, the calculation time for the curve *5* exceeds 30 min, while the calculation by the proposed modified SH algorithm requires 5 s.

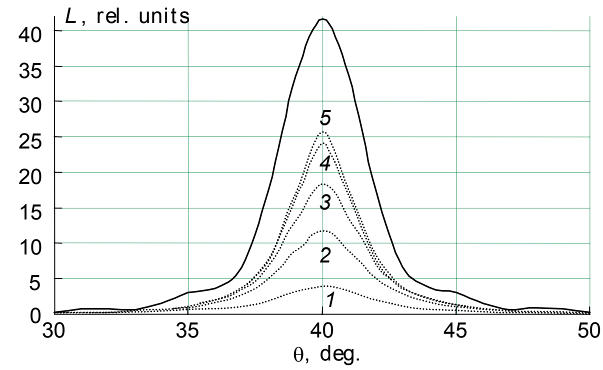

Fig. 4. Brightness of radiation passing through the layer at $\tau_0 = 20.0$, $\Lambda = 0.8$, $\theta_0 = 40°$. In calculations by the present algorithm $N = 99$, $M = 4$. Dotted lines show the calculations by MSH (Ref. 3): $N = 99$, $M = 4$ (*1*); $N = 399$, $M = 16$ (*2*); $N = 399$, $M = 32$ (*3*); $N = 399$, $M = 64$ (*4*); $N = 399$, $M = 128$ (*5*).

Note for the conclusion that the proposed algorithm considered for the case of the plane medium will have much greater importance in the case of concentrated sources, where the hyperbolic and logarithmic peculiarities are contained in the first and second multiplicities[13] of direct radiation in addition to the δ-peculiarity, and the small-angle modification of the MSH accurately takes them into account.[14]

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