SOLUTION OF THE VECTOR RADIATIVE TRANSFER EQUATION IN THE SMALL-ANGLE APPROXIMATION OF THE SPHERICAL HARMONICS METHOD

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In this paper we present a version of the small–angle approximation of the spherical harmonics method for solving the radiation transfer equation in vector form. The approach we propose in this paper allows analytical simple description of the main effects accompanying the propagation of a polarized radiation through turbid media. The accuracy of this technique is estimated, and it is compared with that of the exact solutions of the radiation transfer equation.

All the information available from optical remote sensing of natural formations is in the spatial-angular and spectral distributions of the characteristics of polarization of radiation reflected and scattered by such formations.[1] Following the photometric (ray) description of radiation its state of polarization is defined by a four-dimensional vector. The components of such a vector are the coefficients of the coherence matrix representation in a corresponding basis.[1] Most often used representation[1] is the so-called SP-representation of polarization (i.e., Stokes polarization) via four easily measurable parameters $\mathbf{L} = \{I, Q, U, V\}$. Within the geometric optics approximation the transformation of polarized radiation by a medium is found from the corresponding boundary–value problem formulated for the vector equation of radiation transfer (VERT).[2,3]

The difficulties of solving VERT are well-known[4]: even with the state-of-the-art numerical algorithms and all the computer power available, solution of polarization problems requires many hours of computations. Solution of this problem for the case of real media is more complicated owing the angular anisotropy of scattering (i.e., matrices of the aerosol type). As in the scalar case, the small-angle approximation was also developed for this case.[5–7] However, the results obtained describe extinction of the components of vector-parameter in the medium, but neither yield their mutual transformations, nor (that is particularly important for the tasks of optical sensing) describe the generation of one or another state of polarization in the medium. Such techniques were developed in their vector version on the basis of the scalar small-angle approximation[8,9] that follows from the more general form.[10] In this paper we propose further development of the ideas of small-angle approximation[10] as applied to solving VERT.

Among the tasks of passive remote sensing those, in which the sensed volume is strongly extended along one of its dimensions, are of most practical interest. The cases of natural illumination of the Earth's surface, limited atmospheric or sea layers, etc. fall in this category. To describe mathematically radiation transfer in such media, we use the model of a plane-parallel turbid layer. Decomposition of the relevant boundary problems shows[3] that the general boundary–value problem is then reduced to solution of VERT for a turbid medium layer with a plane wave of arbitrarily polarized light incident upon it

$$\mu \frac{\partial}{\partial t} \mathbf{L}(z,\hat{\mathbf{l}}) + \varepsilon(z)\, \mathbf{L}(z,\hat{\mathbf{l}}) = \frac{\sigma(z)}{4\pi} \oint \overleftrightarrow{S}(\hat{\mathbf{l}},\hat{\mathbf{l}}')\, \mathbf{L}(z,\hat{\mathbf{l}}')\, d\hat{\mathbf{l}}', \quad (1)$$

$$\begin{cases} \mathbf{L}(z,\hat{\mathbf{l}})\big|_{\Gamma_1} = \boldsymbol{L}_0\, \delta(\hat{\mathbf{l}} - \hat{\mathbf{l}}_0), \\ \mathbf{L}(z,\hat{\mathbf{l}})\big|_{\Gamma_2} = 0; \end{cases}$$

where $\mathbf{L}(z, \hat{\mathbf{l}})$ is the vector-parameter of polarization of light along the direction $\hat{\mathbf{l}} = \{\mu, \sqrt{1-\mu^2}\cos\varphi, \sqrt{1-\mu^2}\sin\varphi\}$ at depth z within the medium; $OXYZ$ is the Cartesian system of coordinates; $\mu = (\hat{\mathbf{z}}, \hat{\mathbf{l}}) = \cos\Theta$; Θ and φ are the zenith and the azimuth angle, respectively; $\hat{\mathbf{z}}$ is the unit vector along the direction OZ; $\hat{\mathbf{l}}_0 = \{\zeta, \sqrt{1-\zeta^2}, 0\}$ is the direction of incidence of solar radiation; $\mu_0 = (\hat{\mathbf{z}}, \hat{\mathbf{l}}_0) = \cos\Theta_0$; Θ_0 is the solar zenith angle; $\Gamma_1 = \{z = 0, \hat{\mathbf{l}} \in \Omega_+\}$, $\Gamma_2 = \{z = H, \hat{\mathbf{l}} \in \Omega_-\}$; Ω is the unit sphere: $\Omega = \Omega_+ \cup \Omega_-$, Ω_+ $(\mu>0)$, Ω_- $(\mu<0)$, are low and the upper hemispheres. The medium is characterized by the following parameters: $\varepsilon(z)$ is extinction coefficient; $\Lambda(z)$ is single scattering albedo; $\sigma(z) = \Lambda(z)\,\varepsilon(z)$ is scattering coefficient; $\overleftrightarrow{x}(\cos\gamma)$ is scattering phase matrix, $\cos\gamma = (\hat{\mathbf{l}}, \hat{\mathbf{l}}')$ (the symbol ^ hereinafter denotes a unit vector, and ↔ denotes a matrix).

$\overleftrightarrow{S}(\hat{\mathbf{l}}, \hat{\mathbf{l}}') = \overleftrightarrow{\mathbf{R}}(\hat{\mathbf{l}} \times \hat{\mathbf{l}}' \to \hat{\mathbf{l}} \times \hat{\mathbf{l}}_0)\, \overleftrightarrow{x}(\hat{\mathbf{l}}, \hat{\mathbf{l}}')\, \overleftrightarrow{\mathbf{R}}(\hat{\mathbf{l}}_0 \times \hat{\mathbf{l}}' \to \hat{\mathbf{l}} \times \hat{\mathbf{l}}')$ is the matrix of local transformation due to scattering[1]; $\overleftrightarrow{\mathbf{R}}(\hat{\mathbf{l}} \times \hat{\mathbf{l}}' \to \hat{\mathbf{l}} \times \hat{\mathbf{l}}_0) = \overleftrightarrow{\mathbf{R}}(\chi)$ is the matrix of transformation (rotator) of the vector-parameter due to rotation of the reference plane, where χ is the dihedral angle between the planes $(\hat{\mathbf{l}} \times \hat{\mathbf{l}}')$ and $(\hat{\mathbf{l}} \times \hat{\mathbf{l}}_0)$ (see Refs. 1–3). Further, τ is the current optical depth; τ_0 is the optical thickness of the layer. Incident radiation is characterized by its vector-parameter $\mathbf{L}_0$.

Approximation[10] is constructed in its scalar version based on the small-angle modification of the technique of spherical harmonics (SH). The severity of using SH in the vector case, as well as of solving the boundary–value problem, in general, is due to the form of the matrix of local transformation $\overleftrightarrow{S}$. For a preset $\overleftrightarrow{x}(\cos\gamma)$, the latter depends, in its turn, on the form of the rotator $\overleftrightarrow{\mathbf{R}}$. The form of the rotator matrix depends on the form of presentation of the radiation polarization. A particularly simple (diagonal) form of $\overleftrightarrow{\mathbf{R}}$ is reached in the case when the matrix of coherence is expanded into the circular basis (that is, in the case of CP presentation (circular polarization))[3,11]

$$\overleftrightarrow{\mathbf{R}}(\chi) = \mathrm{Diag}\,(e^{i2\chi},\ 1,\ 1,\ e^{-i2\chi}), \quad (2)$$

Moreover, the SP and the CP presentations are related to each other by a linear expression

$$\mathbf{L}_{\mathrm{CP}} = \begin{bmatrix} L_{+2} \\ L_{+0} \\ L_{-0} \\ L_{-2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} Q-iU \\ I-V \\ I+V \\ Q+iU \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 1 & -i & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \end{bmatrix}\begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} =$$

$$= \overleftrightarrow{\mathbf{T}}_{\mathrm{CS}}\,\mathbf{L}_{\mathrm{SP}}\ , \quad (3)$$

where $\overleftrightarrow{\mathbf{T}}_{\mathrm{CS}}$ is the matrix of transition; i is the imaginary unit.

However, such a transition also transforms the scattering phase matrix

$$\overleftrightarrow{x}_{\mathrm{CP}}(\hat{\mathbf{l}},\ \hat{\mathbf{l}}') = \overleftrightarrow{\mathbf{T}}_{\mathrm{CS}}\ \overleftrightarrow{x}_{\mathrm{SP}}(\hat{\mathbf{l}},\ \hat{\mathbf{l}}')\ \overleftrightarrow{\mathbf{T}}_{\mathrm{CS}}^{-1}, \quad (4)$$

so that, in agreement with Eq. (4), we have for the case of aerosol scattering[12]

$$\overleftrightarrow{x}_{\mathrm{SP}}(\gamma) = \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ a_2 & a_1 & 0 & 0 \\ 0 & 0 & a_3 & a_4 \\ 0 & 0 & -a_4 & a_3 \end{bmatrix},$$

$$\overleftrightarrow{x}_{\mathrm{CP}}(\gamma) = \frac{1}{2}\begin{bmatrix} a_1+a_3 & a_2+ia_4 & a_2-ia_4 & a_1-a_3 \\ a_2+ia_4 & a_1+a_3 & a_1-a_3 & a_2-ia_4 \\ a_2-ia_4 & a_1-a_3 & a_1+a_3 & a_2+ia_4 \\ a_1-a_3 & a_2-ia_4 & a_2+ia_4 & a_1+a_3 \end{bmatrix}. \quad (5)$$

To convert to the system of equations corresponding to the SH technique,[11,13] we introduce generalized spherical coordinates

$$\overleftrightarrow{\mathbf{Y}}^n_l(\mu) = \mathrm{Diag}\{P^{\,l}_{n,+2}(\mu),\ P^{\,l}_{n,+0}(\mu),\ P^{\,l}_{n,-0}(\mu),\ P^{\,l}_{n,-2}(\mu)\},$$

where $P^{\,l}_{ns}(\mu)$ are the generalized Legendre polynomials,[14] s = ±0, ±2.

For the generalized polynomials, the following recursion formulas are valid[14]

$$(2l+1)\mu\overleftrightarrow{\mathbf{Y}}^n_l(\mu) = \Big[\ \overleftrightarrow{\mathrm{A}}^n_{l+1}\ \overleftrightarrow{\mathrm{Y}}^n_{l+1} + \overleftrightarrow{\mathrm{A}}^n_l\ \overleftrightarrow{\mathrm{Y}}^n_{l-1}\Big] + n\overleftrightarrow{\mathrm{B}}_l\overleftrightarrow{\mathbf{Y}}^n_l, \quad (6)$$

$$i(2l+1)\sqrt{1-\mu^2}\ \overleftrightarrow{\mathbf{Y}}^n_l(\mu) = \Big[d^-_n\ \overleftrightarrow{\mathbf{C}}_{l+1}\ \overleftrightarrow{\mathbf{Y}}^{n-1}_{l+1} + g^+_n\ \overleftrightarrow{\mathbf{C}}_l\ \overleftrightarrow{\mathbf{Y}}^{n-1}_{l-1}\Big] + $$

$$+ h^+_n\ \overleftrightarrow{\mathrm{B}}_l\ \overleftrightarrow{\mathbf{Y}}^{n-1}_l; \quad (7)$$

$$i(2l+1)\sqrt{1-m^2}\ \overleftrightarrow{\mathbf{Y}}^n_l(\mu) = \Big[\ d^+_n\ \overleftrightarrow{\mathbf{C}}_{l+1}\ \overleftrightarrow{\mathbf{Y}}^{n+1}_{l+1} + g^-_n\ \overleftrightarrow{\mathbf{C}}_l\ \overleftrightarrow{\mathbf{Y}}^{n+1}_{l-1}\Big] + $$

$$+ h^-_n\ \overleftrightarrow{\mathrm{B}}_l\ \overleftrightarrow{\mathbf{Y}}^{n+1}_l \quad (8)$$

as well as the theorem of summation

$$e^{in\chi}\ P^{\,l}_{ns}(\hat{\mathbf{l}}\times\hat{\mathbf{l}}')e^{-is\chi} = \sum_{r=-l}^{r=+l} P^{\,l}_{nr}(\hat{\mathbf{l}}_0\times\hat{\mathbf{l}})\ \overline{P^{\,l}_{rs}\ \hat{\mathbf{l}}_0\times\hat{\mathbf{l}}}\quad e^{ir(\varphi-\varphi')} \quad (9)$$

and the orthogonality relationship[14]

$$\int_{-1}^{+1} \overleftrightarrow{\mathbf{Y}}^n_k(\mu)\ \overline{\hat{\mathrm{Y}}^n_l(\mu)}\ d\mu = \frac{2}{2\,l+1}\ \overleftrightarrow{1}\ \delta_{lk}, \quad (10)$$

where

$$d^{\pm}_n = \sqrt{(l\pm n+1)\,(l\pm n+2)};\ g^{\leftrightarrow}_n = \sqrt{(l\pm n-1)\,(l\pm n)};$$

$$h^{\leftrightarrow}_n = \sqrt{(l\pm n)\,(l\mp n+2)};$$

$$\overleftrightarrow{\mathrm{A}}^n_l = \mathrm{Diag}\left(\frac{\sqrt{(l^2-s^2)\,(l^2-n^2)}}{l}\right);$$

$$\overleftrightarrow{\mathrm{B}}_l = \mathrm{Diag}\left(\frac{(2\,l+1)s}{l(l+1)}\right);\ \overleftrightarrow{\mathbf{C}}_l = \mathrm{Diag}\left(\frac{\sqrt{l^2-s^2}}{l}\right);$$

and, δ_{lk} is the Kronekker symbol; $\overleftrightarrow{1} = \mathrm{Diag}\ \{1,\ 1,\ 1,\ 1\}$. The upper bar denotes complex conjugation.

Let us present all the functions entering into Eq. (1) in the form of series expansions over generalized spherical functions.

$$[\overleftrightarrow{x}_{\mathrm{CP}}\,(\cos\gamma)]_{rs} = \sum_{l=0}^{\infty}(2\,l+1)\ x^l_{rs}\ P^l_{rs}\,(\cos\gamma);$$

$$\mathbf{L}(\tau,\nu,\varphi) = \sum_{n=-\infty}^{+\infty}\ \sum_{l=0}^{\infty}(2\,l+1)/4\pi\ e^{in\varphi}\ \overleftrightarrow{\mathbf{Y}}^n_l(\nu)\ \mathbf{f}^{\,n}_l(\tau), \quad (11)$$

where $\nu = (\hat{\mathbf{l}},\ \hat{\mathbf{l}}_0) = \cos\psi$ is the cosine of the zenith angle, and φ is the azimuth angle of $\hat{\mathbf{l}}$ with respect to $\hat{\mathbf{l}}_0$ direction, that is, to the direction of incidence of radiation; note also that

$$\mu = \nu\mu_0 + \sqrt{1-\mu_0^2}\times\sqrt{1-\nu^2}\ \cos\varphi,\ (\overleftrightarrow{x}_k)_{rs} = x^k_{rs}.$$

Substitute now the expansions (11) into the equation (1), multiply both its sides by $\overleftrightarrow{\mathbf{Y}}^m_k(\mu)\ e^{-im\varphi}$, and integrate over the total solid angle $d\hat{\mathbf{l}}$. Then, accounting for expressions (6)–(10) we obtain for $\overleftrightarrow{\mathbf{Y}}^n_k(\mu)$ a system of coupled differential equations of the SH technique in its application to VERT:

$$\frac{1}{2\,k+1}\ \frac{d}{dt}\Big\{\mu_0\Big[\ \overleftrightarrow{\mathrm{A}}^m_k\ \mathbf{f}^m_{k-1} + \overleftrightarrow{\mathrm{A}}^m_{k+1}\ \mathbf{f}^m_{k+1} + m\ \hat{\mathrm{B}}_k\ \mathbf{f}^m_k\Big]\ -$$

$$-i/2\sqrt{1-\mu_0^2}\ \Big[d^+_{m-1}\ \overleftrightarrow{\mathrm{C}}_k\ \mathbf{f}^{\,m-1}_{k-1}\ - g^-_{m-1}\ \overleftrightarrow{\mathrm{C}}_{k+1}\ \mathbf{f}^{m-1}_{k+1} +$$

$$+ h^-_{m-1}\ \overleftrightarrow{\mathrm{B}}_k\ \mathrm{f}^{m-1}_k + d^-_{m+1}\ \overleftrightarrow{\mathrm{C}}_k\ \mathbf{f}^{m+1}_{k-1} - g^+_{m+1}\ \overleftrightarrow{\mathrm{C}}_{k+1}\ \mathbf{f}^{m+1}_{k+1} +$$

$$+ h^-_{m+1}\ \overleftrightarrow{\mathrm{B}}_k\ \mathbf{f}^{m+1}_k\Big]\Big\} + (1-\Lambda)\ \overleftrightarrow{x}_k\ \mathbf{f}^m_k = 0. \quad (12)$$

The difference of the obtained SH system from that developed in Ref. 13 is that the axis of its system of coordinates for the unit vectors of directions in space follows $\hat{\mathbf{l}}_0$ (see expression (12)), while in Ref. 13 the axis $\hat{\mathbf{z}}$ was taken for such a direction.

In its scalar case[10] the small–angle modification of the SH technique (SASH) is based on the possibility of approximating the coefficients of the Legendre polynomial expansion of the brightness body by a monotonic slowly diminishing function of the running number in the expansion. Then the dependence of such coefficients on the running number may be expanded into a series, and keeping only first two terms in the SH system after the substitution, one may reduce such a system of equations to a single differential equation. Such an approximation is based on a sharply anisotropic shape of the brightness body which follows from the particular features of the Green's function solution of the equation of radiation transfer.[15]

Consider the boundary conditions for the problem (1) in more detail. In case of CP representation of the incident radiation of brightness L, polarization p, ellipticity q (Ref. 1), one may recall expression (3) and write for a plane of polarization coinciding with the plane $(\hat{\mathbf{l}}_0 \times \hat{\mathbf{z}})$, in the reference plane $(\hat{\mathbf{l}}' \times \hat{\mathbf{l}}_0)$, that for radiation propagating into the forward hemisphere

$$\mathbf{L}^0_{\rm CP}(\tau = 0, \hat{\mathbf{l}}) = \frac{L}{2}\begin{bmatrix} p\ e^{i2\varphi} \\ 1-q \\ 1+q \\ p\ e^{-i2\varphi}\end{bmatrix} \delta(\hat{\mathbf{l}} - \hat{\mathbf{l}}_0). \qquad (13)$$

Expanding the upper boundary condition (13) into a series, we obtain

$$\mathbf{L}^0_{\rm CP}(\tau = 0, \hat{\mathbf{l}}) = \frac{L}{2}\sum_{k \ge s}^{\infty} \frac{2k+1}{4\pi}\begin{bmatrix} p\ e^{i2\varphi} P^k_{+2,+2}(\nu) \\ (1-q) P^k_{+0,+0}(\nu) \\ (1+q) P^k_{-0,-0}(\nu) \\ p\ e^{-i2\varphi} P^k_{-2,-2}(\nu)\end{bmatrix}. \qquad (14)$$

As is seen from Eq. (14), all quantities $\mathbf{f}^m_k (\tau = 0) = \text{const}$. With the increase of τ $\mathbf{f}^m_k \to 0$ at $k \to \infty$, due to scattering. However, at small values of τ, when first orders of scattering dominate and aerosol backscattering can be neglected, the dependence of $\mathbf{f}^m_k$ on k is bound to be a slowly monotonically decreasing function. Thus, it becomes possible to introduce a continuous dependence of the $\mathbf{f}^m(k)$ coefficients on the number k, similar to that in the scalar case. It is further seen from Eq. (14) that the azimuthal part only contains $m = 0, \pm 2, \ldots$, that is, $k > m$. Assume that such a relation also holds for $\tau = 0$. We expand $\mathbf{f}^m(k)$ into the Tailor series:

$$\mathbf{f}^m(k \pm 1) = \mathbf{f}^m(k) \pm \frac{\partial\, \mathbf{f}^m(k)}{\partial k} + \ldots\ . \qquad (15)$$

By substituting Eq. (15) into Eq. (12), and truncating the series at the first order terms, we obtain for $k \gg m$ and $\mu_0 \to 1$

$$\mu_0 \frac{d}{d\tau}\mathbf{f}^m(\tau) + (\overleftrightarrow{1} - \Lambda \overleftrightarrow{x}_k)\, \mathbf{f}^m(\tau) = 0. \qquad (16)$$

Since we have neglected aerosol backscattering in the boundary conditions, we have for them in SASH approach the following formulas:

$$\mathbf{f}^0(0) = \frac{L}{2}\begin{bmatrix} 0 \\ 1-q \\ 1+q \\ 0\end{bmatrix}; \qquad \mathbf{f}^2(0) = \frac{L}{2}\begin{bmatrix} p \\ 0 \\ 0 \\ p\end{bmatrix};$$

$$\mathbf{f}^m(\tau) \to 0 \ \text{ for } \tau \to \infty. \qquad (17)$$

The solution of the matrix differential equation (16) under the boundary conditions (17) for a homogeneous with depth turbid medium is[16]

$$\mathbf{f}^m_k(\tau) = \exp(-\tau/\mu_0)\left(\sum_{i=1}^{4} \frac{\exp(\Delta\tau\zeta_i/\mu_0)(\zeta_i \overleftrightarrow{1} - \overleftrightarrow{x}_k)^{\rm v}}{\dfrac{d}{dz}\det(z\overleftrightarrow{1} - \overleftrightarrow{x}_k)\Big|_{z=\zeta_i}}\right)\mathbf{f}^m_k(0),$$

(18)

where the symbol v denotes the matrix of cofactors to a given matrix, and ζ_i are solutions of the characteristic equation

$$\det(\zeta\overleftrightarrow{1} - \overleftrightarrow{x}_k) = 0. \qquad (19)$$

Just the expression (18) is the solution to VERT sought obtained within the small–angle modification of the SH technique. It is easy to see that in this approximation $\mathbf{f}^m_k(\tau)$ does not actually depend on the azimuth index m, i.e., the azimuthal spectrum of incident radiation keeps with depth. Since the incident radiation only contains the harmonics with $m = 0, \pm 2$, the solution of the boundary–value problem can be presented in the form:

$$\mathbf{L}(\tau, \nu, \varphi) = \sum_{n=-1}^{+1}\sum_{l=0}^{\infty} \frac{2l+1}{4\pi}\, e^{i2n\varphi}\, \overleftrightarrow{\mathbf{Y}}^{2n}_l(\nu)\, \overleftrightarrow{\Gamma}_l(\tau)\, \mathbf{f}^{2n}_l(0), \quad (20)$$

where

$$\overleftrightarrow{\Gamma}_k(\tau) = \sum_{i=1}^{4} \frac{\exp(\Delta\tau\zeta_i/\mu_0)\,(\zeta_i\overleftrightarrow{1} - \overleftrightarrow{x}_k)^{\rm v}}{\dfrac{d}{dz}\det(z\overleftrightarrow{1} - \overleftrightarrow{x}_k)\Big|_{z=f_i}} \exp(-\tau/\mu_0) \quad (21)$$

is the coefficient of expansion of the matrix Green's function for plane layer of a turbid medium (with the transfer matrix from Ref. 1).

The properties of the transfer matrix (21) are determined by the roots of the characteristic equation, dependent, in their turn, on the matrix $\overleftrightarrow{x}_k$. From the expansion (11) and formula (5) we have

$$\overleftrightarrow{x}_k = \begin{bmatrix} u_k & c_k & \bar{c}_k & v_k \\ c_k & x_k & s_k & \bar{c}_k \\ \bar{c}_k & s_k & x_k & c_k \\ v_k & \bar{c}_k & c_k & u_k \end{bmatrix}, \qquad (22)$$

where

$$u_k = \frac{1}{2}\int_{-1}^{+1} \frac{a_1(\mu) + a_3(\mu)}{2} P^k_{2,2}(\mu)\, d\mu,$$

$$v_k = \frac{1}{2}\int_{-1}^{+1} \frac{a_1(\mu) - a_3(\mu)}{2} P^k_{2,-2}(\mu)\, d\mu,$$

$$x_k = \frac{1}{2}\int_{-1}^{+1} \frac{a_1(\mu) + a_3(\mu)}{2} P^k_{0,0}(\mu)\, d\mu,$$

$$s_k = \frac{1}{2}\int_{-1}^{+1} \frac{a_1(\mu) - a_3(\mu)}{2} P^k_{0,0}(\mu)\, d\mu,$$

$$c_k = \frac{1}{2}\int_{-1}^{+1} \frac{a_2(\mu) + i a_4(\mu)}{2} P^k_{2,0}(\mu)\, d\mu.$$

Because of a strong anisotropy of the scattering phase function (weak backscattering) and according to the following property[12] $a_3(\mu) \approx a_1(\mu)$ at $\mu \to 1$, $a_3(\mu) \approx -a_1(\mu)$ at $\mu \to -1$, valid for aerosol particles, the values of coefficient s_k and ν_k are negligible.

As a consequence, solutions of the characteristic equation take the form

$$\zeta_{12,34} = (x_k + u_k)\,/\,2 \pm \sqrt{\frac{(x_k - u_k)^2}{4} + 4\,r^2} = \alpha + \Lambda, \quad (23)$$

where

$$r = \begin{cases} \mathrm{Re}\; c_k = P, & \text{for roots 1 and 2} \\ i\;\mathrm{Im}\; c_k = iQ, & \text{for roots 3 and 4} \end{cases}$$

$$\Delta = \begin{cases} \pm\,\beta, & \text{for roots 1 and 2} \\ \pm\, i\delta, & \text{for roots 3 and 4;} \end{cases}$$

For simplicity hereinafter index k is omitted in the expressions for the coefficients ζ, r, α, Δ, P, Q, β, and δ. To avoide ambiguity, we will consider the value δ to be a real value in all expressions below.

By substituting the values of roots ζ into the expressions (17), (20), and (21), and taking into account the relation between the SP and the CP representations, we obtain:

$$\mathbf{L}_{\mathrm{SP}}(\tau,\nu,\varphi) = \frac{L}{2}\exp\left(-\,(1 - \Lambda\alpha)\,\tau\right)\sum_{k=0}^{\infty}\frac{2k+1}{4\pi}\begin{bmatrix} (\cosh(\Delta\beta\tau) + \frac{(\alpha - u)}{\beta}\sinh(\Delta\beta\tau))\,P^k_{0,0}(\nu) + 2pP\beta^{-1}\sinh(\Delta\beta\tau)\cos 2\varphi P^k_{2,0}(\nu) \\ 2P\beta^{-1}\sinh(\Delta\beta\tau)P^k_{2,0}(\nu) + p(\cosh(\Delta\beta\tau) + \frac{(u-\alpha)}{\beta}\sinh(\Delta\beta\tau))\cos 2\varphi\; P^k_{2,2}(\nu) \\ -\,2Qq\delta^{-1}\sin(\Delta\beta\tau)\;P^k_{2,0}(\mathrm{n}) + p(\cos(\mathrm{Ldt}) + \frac{(u-\alpha)}{\delta}\sin(\Delta\beta\tau))\sin 2\varphi\; P^k_{2,2}(\nu) \\ q(\delta\cos(\Delta\beta\tau) + \frac{(\mathrm{a} - u)}{\delta}\sin(\Delta\beta\tau))\,P^k_{0,0}(\nu) + -\,2\,pQ\delta^{-1}\sin(\Delta\beta\tau)\sin 2\varphi\; P^k_{2,0}(\nu) \end{bmatrix}. \quad (24)$$

In order to use these expressions in computations, one needs to have an expansion of $\overset{\leftrightarrow}{x}\,(\hat{\mathbf{l}}, \hat{\mathbf{l}}')$ over the generalized spherical functions. These problems were considered by several authors (see, for example, Ref. 17), who gave the computational algorithms and presented examples of such expansions. However, the principal SASH assumption, formulated above on the slowly monotonically decreasing coefficients of the series expansion of the vector-parameter of polarization and the neglect of backscattering impose certain restrictions on the nature of scattering phase matrix. Since characteristics of scattered radiation are mainly described by the scattering phase matrix[15] in the case of a plane–parallel scattering layer, the dependence of $\overset{\leftrightarrow}{x}_k$ on k is bound to be a slowly monotonically decreasing function too. However, as it follows from results, obtained in Ref. 17, this restriction is not so strict for aerosol media though the k–behavior of the $\overset{\leftrightarrow}{x}_k$ matrix should be smoothed.

As in the scalar case, it is quite convenient for making computations to approximate an arbitrary scattering phase matrix by a linear combination of matrices in a small–angle approximation written using Henji–Greenstein function[18]

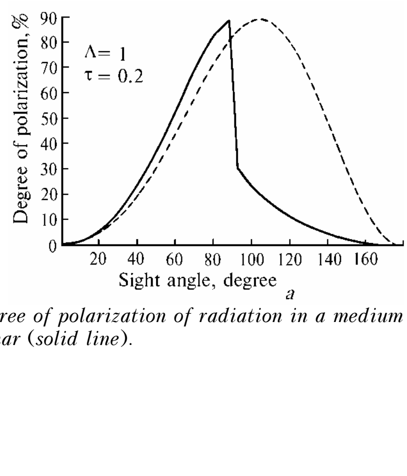
$$\overset{\leftrightarrow}{x}_{\mathrm{HG}}(\mu) = \frac{1 - g^2}{\sqrt{(1 + g^2 - 2g\mu)^3}}\,\overset{\leftrightarrow}{x}_0(\mu)\,, \quad (25)$$

where g is the average cosine of the Henji-Greenstein scattering phase function

$$\overset{\leftrightarrow}{x}_0(\mu) = \begin{bmatrix} 0.25(1+\mu)^2 & 0.5(1-\mu^2)\,R_m & 0.5(1-\mu^2)\,\bar{R}_m & 0 \\ 0.5(1-\mu^2)\,R_m & 1 & 0 & 0.5(1-\mu^2)\,\bar{R}_m \\ 0.5(1-\mu^2)\,\bar{R}_m & 0 & 1 & 0.5(1-\mu^2)\,R_m \\ 0 & 0.5(1-\mu^2)\,\bar{R}_m & 0.5(1-\mu^2)\,R_m & 0.25(1+\mu)^2 \end{bmatrix}.$$

$R_m = P_m + iQ_m$, P_m, and Q_m are the maximum values of the degree of linear and circular polarizations of single scattered radiation. Simple analytical expressions for coefficients of a Henji–Greenstein scattering phase matrix (see Eq. (22)) can be found in Ref. 18.

Non-trivial character of SASH transformations complicates analytical estimation of the accuracy of obtained approximate solution. Most efficient way of doing such an estimation is to compare the approximate solution to the exact numerical or analytical solution for VERT. Figure 1 shows the above obtained solution in comparison with the analytical one obtained by Chandrasekhar[2] for a homogeneous layer with a Rayleigh scattering phase matrix and $\Lambda = 1$. As seen from this figure, the difference between the degrees of linear polarization obtained by the exact technique and SASH is below 10% for optical depths $0.2 < \tau \le 5$ and sighting angles $0 \le \gamma \le 90°$. However, in the upper hemisphere this difference is too large, because of the neglect of the variance of scattered photons paths and backscattering in the small–angle approximation. In fact, the variance of scattered photons paths can not be neglected at $\tau > 5$. This is clearly seen in Fig. 2 which illustrates the decrease of maximum value of the degree of linear polarization with increasing optical thickness of a medium.



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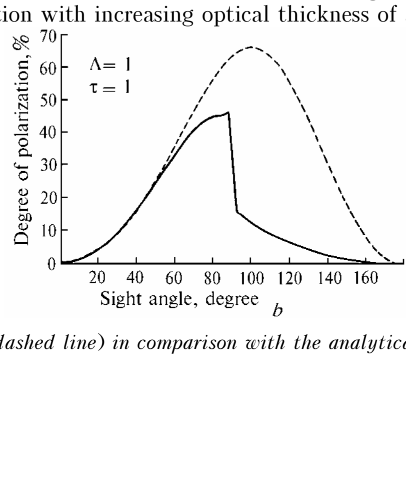

b

Fig. 1. Degree of polarization of radiation in a medium calculated using SASH (dashed line) in comparison with the analytical solution by Chandrasekhar (solid line).

In the case of the aerosol media and non–conservative scattering ($\Lambda \neq 1$) an estimation of the approach proposed was performed by comparing with the results obtained using Monte Carlo method. Results of such computations ($3\cdot 10^{-5}$) trials, the maximum variance within 4%) are presented in Fig. 3. The curves in this figure present calculational results for the forward scattering hemisphere for an aerosol medium with Λ = 0.8 and scattering phase matrix given by formulas (24) and (25) with g = 0.9.

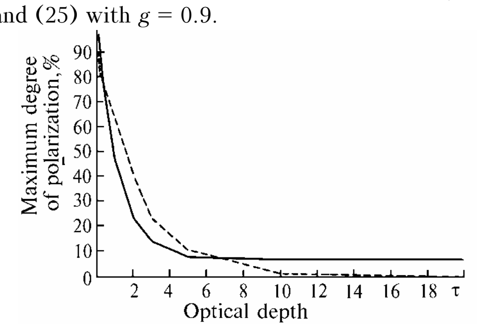

Fig. 2. Maximum degree of polarization as a function of optical depth: SASH (dashed line) and analytical solution by Chandrasekhar (solid line).

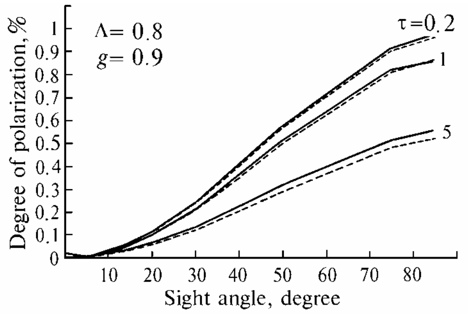

Fig.3. Polarization degree within the aerosol medium and SASH (dashed line) and numerical Monte Carlo solution (solid line).

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