Matrix Form of VRTE Solution for Vertically Stratified Slab

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Abstract— The physical basis of the radiative transfer theory is the ray approximation that causes spatial singularities in the solution of the vectorial radiative transfer equation (VRTE). It is possible to formulate an equation upon the analysis of the angular spectrum of the Stokes vector — the modification of the spherical harmonics method (MSH). The MSH is an approximate solution of the VRTE that includes the solution singularities together with the anisotropic part. The source function built upon the MSH does not change the form of the VRTE boundary problem for the regular smooth part. The regular part of the boundary problem is solved using the discrete ordinates method. The same method allows to obtain exact analytical solutions for the discretized VRTE in the matrix form and to describe the vertical heterogeneity of the slab through the division of the complete radiation stream upon the descending and ascending ones and thus to include the symmetry of the VRTE boundary problem in case of the plane unidirectional source.

1. INTRODUCTION

For the interpretation of results the increased measurement precision of the optical remote sensing systems demands the development of the solution methods of the radiative transfer equation (RTE) without any prior limitations on the medium properties taking into account the polarization of radiation [1]. Since the measurements are carried out for the huge number of spectral ranges (hyperspectral systems) one of the major requests to the RTE solution algorithms is the computational speed.

All the numerical methods of RTE solution are based on the substitution of the scattering integral by the final sum [2]. The ray approximation underlying RTE generates inevitably the singularities in the spatially — angular radiance distribution: any break in the boundary conditions spreads deeply into the medium. The presence of singularities in the radiance angular distribution doesn't allow conducting the scattering integral replacement by the final sum [3]. For the angular δ-singularity elimination Chandrasekhar offered [4] to subtract the direct non-scattered radiation and to state the boundary value problem for the diffused radiation. However all the scattering media (let it be the atmosphere or the ocean) have the suspended particles with the size much larger than light wavelength that according to G. Mie theory gives strong anisotropic light scattering on them. In these conditions the scattered radiation is indistinguishable from the direct radiation and Chandrasekhar method loses its effectiveness. In the series of articles the procedure of the phase scattering function smoothing is offered [2], however it is equal to the neglect of the coarse aerosol fraction from analysis. This problem becomes especially nagging at transferring to RTE solution for the media with 3D geometry, where the radiance angular distribution has the hyperbolic and logarithmic singularities in the first two orders of scattering along with the δ-singularity in the direct radiation [5].

In the paper [6] the different solution approach of the singularities elimination problem in the radiance spatially — angular distribution or in the more general case of the polarized radiation of Stokes vector is offered. The approach [6] is based on the derivation of the approximated equation describing the solution anisotropic part, including all the singularities, and on the statement of the boundary value problem for the rest smooth regular part. The simplest way to formulate the separation of the solution anisotropic part is analyzing the radiance angular spectrum, which should be monotonically slowly descending for the anisotropic component.

2. SOLUTION ANISOTROPIC PART

Let's consider the boundary value problem of the vectorial radiative transfer equation (VRTE) for the turbid medium slab irradiated from above by a plane unidirectional (PU) source of light with arbitrary polarization

$$\begin{cases} \mu\frac{\partial}{\partial\tau}\vec{L}\left(\tau,\hat{\mathbf{l}}\right)+\vec{L}\left(\tau,\hat{\mathbf{l}}\right)=\frac{\Lambda}{4\pi}\oint\overleftrightarrow{R}\left(\hat{\mathbf{l}}\times\hat{\mathbf{l}}'\to\hat{\mathbf{z}}\times\hat{\mathbf{l}}\right)\overleftrightarrow{x}\left(\hat{\mathbf{l}},\hat{\mathbf{l}}'\right)\overleftrightarrow{R}\left(\hat{\mathbf{z}}\times\hat{\mathbf{l}}\to\hat{\mathbf{l}}\times\hat{\mathbf{l}}'\right)\vec{L}\left(\tau,\hat{\mathbf{l}}'\right)d\hat{\mathbf{l}}'; \\ \vec{L}(\tau,\mu,\varphi)\Big|_{\tau=0,\ \mu>0}=\vec{L}_0\delta\left(\hat{\mathbf{l}}-\hat{\mathbf{l}}_0\right);\ \ \vec{L}(\tau,\mu,\varphi)\Big|_{\tau=\tau_0,\ \mu<0}=\vec{0}; \end{cases} \tag{1}$$

where $\vec{L}(\tau,\hat{\mathbf{l}})$ is the Stokes vector at the optical depth τ in the sighting direction $\hat{\mathbf{l}}$; $\overleftrightarrow{x}(\hat{\mathbf{l}},\hat{\mathbf{l}}')$ is the scattering phase matrix, Λ is the single scattering albedo. The unit vector is marked by the symbol hut $\ll\hat{}\gg$. The equation is written in Cartesian coordinates system $OXYZ$, the origin O of which is located on the upper slab boundary, and the axe OZ is directed downwards perpendicularly to the boundary; $\mathbf{r}=x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}$ is the radius-vector of the medium point; $\tau=\varepsilon z$ is the optical depth, ε is the attenuation coefficient, τ_0 is the slab optical thickness. $\hat{\mathbf{l}}=\{\sqrt{1-\mu^2}\cos\varphi,\ \sqrt{1-\mu^2}\sin\varphi,\ \mu\}$, $\hat{\mathbf{l}}_0=\{\sqrt{1-\mu_0^2},0,\ \mu_0\}$ is the direction of the slab irradiation. $\overleftrightarrow{R}(\hat{\mathbf{l}}\times\hat{\mathbf{l}}'\to\hat{\mathbf{z}}\times\hat{\mathbf{l}})$ is the matrix of Stokes parameter transformation by the rotation of the reference plane from $\hat{\mathbf{l}}\times\hat{\mathbf{l}}'$ to $\hat{\mathbf{z}}\times\hat{\mathbf{l}}$-rotator. $\vec{L}_0=L_0\,[1,\quad p\cos\alpha,\quad -p\sin\alpha,\quad q]^T$ is the vector of polarization, L_0 is the radiance, p is the degree of polarization, α is the azimuth of polarization, q is the ellipticity of the incident radiation, the upper index T means the transposition. Hereafter we will designate the column-vector by the symbol $\ll\rightarrow\gg$, the row-vector by the symbol $\ll\leftarrow\gg$, and the matrix by the symbol $\ll\leftrightarrow\gg$.

For the statement of the equation for the anisotropic part we will transfer to the radiance angular spectrum. Let's represent the angular distribution of Stokes parameters and the phase matrix in the form of the series on the generalized surface harmonics in the circular polarization presentation (CP-presentation) [6, 7]:

$$\vec{L}\left(\tau,\hat{\mathbf{l}}\right)=\sum_{m=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{2k+1}{4\pi}\overleftrightarrow{Y}_m^k(\nu)\vec{f}_m^k(\tau)\exp(-im\varphi),\ \ \left[\overleftrightarrow{x}\left(\hat{\mathbf{l}},\hat{\mathbf{l}}'\right)\right]_{r,s}=\sum_{k=0}^{\infty}(2k+1)x_{r,s}^k(\tau)P_{r,s}^k\left(\hat{\mathbf{l}}\cdot\hat{\mathbf{l}}'\right), \tag{2}$$

that reduces VRTE to the infinite ordinary differential equation set [6]:

$$\begin{aligned} &\frac{1}{2k+1}\frac{\partial}{\partial\tau}\left\{\mu_0\left[\overleftrightarrow{A}_m^{k+1}\ \vec{f}_m^{k+1}(\tau)+\overleftrightarrow{B}_m^k\vec{f}_m^k(\tau)+\overleftrightarrow{A}_m^k\vec{f}_m^{k-1}(\tau)\right]+\frac{i}{2}\sqrt{1-\mu_0^2}\right. \\ &\left.\times\left[\overleftrightarrow{h}_1\vec{f}_{m-1}^{k-1}(\tau)+\overleftrightarrow{h}_2\vec{f}_{m-1}^k(\tau)+\overleftrightarrow{h}_3\vec{f}_{m-1}^{k+1}(\tau)+\overleftrightarrow{h}_4\vec{f}_{m+1}^{k-1}(\tau)+\overleftrightarrow{h}_5\vec{f}_{m+1}^k(\tau)+\overleftrightarrow{h}_6\vec{f}_{m+1}^{k+1}(\tau)\right]\right\} \\ &+\left(\overleftrightarrow{1}-\Lambda\overleftrightarrow{x}^k\right)\vec{f}_m^k(\tau)=\vec{0}, \end{aligned} \tag{3}$$

where $\overleftrightarrow{Y}_m^k(\nu)=\mathrm{diag}[P_{m,+2}^k(\nu)\ \ P_{m,+0}^k(\nu)\ \ P_{m,-0}^k(\nu)\ \ P_{m,-2}^k(\nu)]$ are the generalized surface harmonics in the matrix form, $\nu=(\hat{\mathbf{l}},\hat{\mathbf{l}}_0)$, $\overleftrightarrow{A}_m^k$, $\overleftrightarrow{B}_m^k$, $\overleftrightarrow{h}_j$ are the matrices depending only on the indices k and m.

For the equation determination of the anisotropic part let's make the following assumptions:

1. introducing the continuous dependence $\vec{f}^m(k,\tau)$ on the index k, which in integer points coincides with values of the expansion coefficients $\vec{f}_m^k(\tau)$;

2. at a strong anisotropy of the angular distribution of Stokes parameters its spectrum $\vec{f}^m(k,\tau)$ is a slowly monotonically decreasing function of the index k,that allows to expand it in a Taylor series preserving the first two terms

$$\vec{f}^m(\tau,k\pm1)\approx\vec{f}^m(\tau,k)\pm\frac{\partial\vec{f}^m(\tau,k)}{\partial k}; \tag{4}$$

3. owing to the anisotropy of the radiance angular distribution the basic contribution to the solution is given by the terms with $k\gg1$ and its anisotropy is much greater than its asymmetry $k\gg m$.

These assumptions allow reducing the infinite ordinary differential equation set (3) to one partial equation

$$\mu_0\frac{\partial\vec{f}^m}{\partial\tau}+\frac{\sqrt{1-\mu_0^2}}{2}\frac{\partial}{\partial\tau}\left[-\left(\frac{\partial\vec{f}^{m-1}}{\partial\kappa}+\frac{\partial\vec{f}^{m+1}}{\partial\kappa}\right)+\frac{m}{\kappa}\left(\vec{f}^{m-1}-\vec{f}^{m+1}\right)\right]=-\left(\overleftrightarrow{1}-\Lambda\overleftrightarrow{x}^k\right)\vec{f}^m(\tau,\kappa), \quad (5)$$

where $\kappa = \sqrt{k(k+1)}$. The items of the order κ^{-1} were eliminated by the derivation of the Equation (5).

Let's introduce the function $\vec{\omega} = \vec{\omega}(\tau,\psi,k)$, the azimuthal spectrum of which is the unknown expansion coefficient of the Stokes parameters angular distribution

$$\vec{\omega}(\tau,\psi,\kappa)=\sum_{m=-\infty}^{\infty}\vec{f}^m(\tau,\kappa)\exp(im\psi);\quad \vec{f}^m(\tau,\kappa)=\frac{1}{2\pi}\int_0^{2\pi}\vec{\omega}(\tau,\psi,\kappa)\exp(-im\psi)\,d\psi, \quad (6)$$

that reduce to the equation

$$\frac{\partial}{\partial\tau}\left[\mu_0-i\left(\hat{\mathbf{l}}_{0\perp},\nabla_\kappa\right)\right]\vec{\omega}(\tau,\psi,\kappa)=-\left(\overleftrightarrow{1}-\Lambda(\tau)\overleftrightarrow{x}^k(\tau)\right)\vec{\omega}(\tau,\psi,\kappa). \quad (7)$$

The received equation describes the angular spectrum taking into account the mesh of azimuthal harmonics that allows describing the rotation of the radiance angular distribution maximum from the incident direction near the boundary to the vertical direction in the medium depth. Since our concern is only the anisotropic part near the slab boundary, we can neglect the second item in the approximation of the small incident angle $\sqrt{1-\mu_0^2}\to 0$ that permits to get the analytically simple solution of the Equation (7)

$$\vec{f}^m(\tau,k)=\exp\left\{-\tau(\overleftrightarrow{1}-\Lambda\overleftrightarrow{x}^k)\big/\mu_0\right\}\vec{f}^m(0,k)=\overleftrightarrow{Z}_k(\tau)\vec{f}^m(0,k). \quad (8)$$

Substituting the boundary conditions and returning the Stokes polarization presentation (SP-presentation) we will get the expression for the solution regular part:

$$\begin{aligned}\vec{L}_a\left(\tau,\hat{\mathbf{l}}\right) &= L_0\sum_{k=0}^{\infty}\frac{2k+1}{2\pi}\left\{\overleftrightarrow{P}_R^{k,0}(\nu)\overleftrightarrow{Z}_k(\tau)\begin{bmatrix}1\\0\\0\\q\end{bmatrix}-\overleftrightarrow{P}_R^{k,2}(\nu)\overleftrightarrow{Z}_k(\tau)\begin{bmatrix}0\\p\cos 2\varphi\\p\sin 2\varphi\\0\end{bmatrix}\right.\\ &\left.-\overleftrightarrow{P}_I^{k,2}(\nu)\overleftrightarrow{Z}_k(\tau)\begin{bmatrix}0\\p\sin 2\varphi\\p\cos 2\varphi\\0\end{bmatrix}\right\},\end{aligned} \quad (9)$$

$$\overleftrightarrow{P}_R^{k,m}(\mu)=\begin{bmatrix}Q_k^m&0&0&0\\0&R_k^m&0&0\\0&0&R_k^m&0\\0&0&0&Q_k^m\end{bmatrix},\ \overleftrightarrow{P}_I^{k,m}(\mu)=\begin{bmatrix}0&0&0&0\\0&0&T_k^m&0\\0&-T_k^m&0&0\\0&0&0&0\end{bmatrix},\ \begin{aligned}&Q_k^m(\mu)=P_{m,0}^k(\mu),\\&R_k^m(\mu)=0.5i^m\left(P_{m,+2}^k(\mu)+P_{m,-2}^k(\mu)\right),\\&T_k^m(\mu)=0.5i^m\left(P_{m,+2}^k(\mu)-P_{m,-2}^k(\mu)\right).\end{aligned}$$

3. SOLUTION REGULAR PART

Now let's consider the smooth part of Stokes parameters field in the slab irradiated from above by a PU-source (1). In this case we have VRTE

$$\mu\frac{\partial}{\partial\tau}\tilde{\vec{L}}\left(\tau,\hat{\mathbf{l}}\right)+\tilde{\vec{L}}\left(\tau,\hat{\mathbf{l}}\right)=\frac{\Lambda}{4\pi}\oint\overleftrightarrow{R}\left(\hat{\mathbf{l}}\times\hat{\mathbf{l}}'\to\hat{\mathbf{l}}\times\hat{\mathbf{l}}_0\right)\overleftrightarrow{x}\left(\hat{\mathbf{l}},\hat{\mathbf{l}}'\right)\overleftrightarrow{R}\left(\hat{\mathbf{l}}_0\times\hat{\mathbf{l}}'\to\hat{\mathbf{l}}\times\hat{\mathbf{l}}'\right)\tilde{\vec{L}}\left(\tau,\hat{\mathbf{l}}'\right)d\hat{\mathbf{l}}'+\vec{\Delta}\left(\tau,\hat{\mathbf{l}}\right), \quad (10)$$

where the complete solution is presented by the expression [6]

$$\vec{L}(\tau,\mu,\varphi)=\tilde{\vec{L}}(\tau,\mu,\varphi)+\vec{L}_a(\tau,\mu,\varphi), \quad (11)$$

and the source function is

$$\vec{\Delta}\left(\tau,\hat{\mathbf{l}}\right)=\frac{\Lambda}{4\pi}\oint\overleftrightarrow{R}\left(\hat{\mathbf{l}}\times\hat{\mathbf{l}}'\to\hat{\mathbf{l}}\times\hat{\mathbf{l}}_0\right)\overleftrightarrow{x}\left(\hat{\mathbf{l}},\hat{\mathbf{l}}'\right)\overleftrightarrow{R}\left(\hat{\mathbf{l}}_0\times\hat{\mathbf{l}}'\to\hat{\mathbf{l}}\times\hat{\mathbf{l}}'\right)\vec{L}_a\left(\tau,\hat{\mathbf{l}}'\right)d\hat{\mathbf{l}}'-\mu\frac{\partial}{\partial\tau}\vec{L}_a\left(\tau,\hat{\mathbf{l}}\right)-\vec{L}_a\left(\tau,\hat{\mathbf{l}}\right). \quad (12)$$

Then we deal with the scattering integral. In CP-presentation all the coefficients in VRTE become complex that makes it difficult to use the effective numerical methods of the VRTE solution that is based on the sorting algorithm. We convert all the functions under the integral to CP-presentation, expand the scattering matrix in series on a generalized spherical harmonics, use the addition theorem and return to SP-presentation. It can be written down as follows:

$$\begin{aligned}\vec{I}_S &= \overleftrightarrow{T}_{SC}\frac{\Lambda}{4\pi}\oint \overleftrightarrow{T}_{CS}\overleftrightarrow{R}(\chi)\overleftrightarrow{T}_{SC}\overleftrightarrow{T}_{CS}\overleftrightarrow{x}\left(\hat{\mathbf{l}},\hat{\mathbf{l}}'\right)\overleftrightarrow{T}_{SC}\overleftrightarrow{T}_{CS}\overleftrightarrow{R}\left(\chi'\right)\overleftrightarrow{T}_{SC}\overleftrightarrow{T}_{CS}\vec{\tilde{L}}\left(z,\hat{\mathbf{l}}'\right)d\hat{\mathbf{l}}' \\ &= \frac{\Lambda}{4\pi}\oint\left(\sum_{k=0}^{\infty}(2k+1)\sum_{m=-k}^{k}e^{im(\varphi-\varphi')}\overleftrightarrow{P}^{k}_{m}(\mu)\overleftrightarrow{\chi}_k\overleftrightarrow{P}^{k}_{m}(\mu')\right)\vec{\tilde{L}}\left(\tau,\hat{\mathbf{l}}'\right)d\hat{\mathbf{l}}', \end{aligned} \tag{13}$$

where $\overleftrightarrow{\chi}_k = \overleftrightarrow{T}_{SC}\overleftrightarrow{x}_k\overleftrightarrow{T}_{CS}$, $\overleftrightarrow{T}_{SC}$, $\overleftrightarrow{T}_{CS}$ are the transfer matrices from SP-presentation to CP and vice-versa accordingly [6].

It is easy to get convinced by the direct verification that $\overleftrightarrow{P}^{k}_{m}(\mu) = \overline{\overleftrightarrow{P}^{k}_{-m}(\mu')}$, where the line above indicates the complex-conjugate number. It means that the local transformation matrix is a real function. Therefore there is no need to keep all the terms in the azimuth series (13) and to combine the terms of the series with m and $-m$. Then we determine the matrices

$$\overleftrightarrow{\phi}_1(\varphi) = \mathrm{diag}\{\cos\varphi,\cos\varphi,\sin\varphi,\sin\varphi\},\quad \overleftrightarrow{\phi}_2(\varphi) = \mathrm{diag}\{-\sin\varphi,-\sin\varphi,\cos\varphi,\cos\varphi\};$$

$$\overleftrightarrow{D}_1=\mathrm{diag}\{1,1,0,0\},\ \overleftrightarrow{D}_2=\mathrm{diag}\{0,0,-1,-1\};\ \overleftrightarrow{\Pi}^{m}_{k}(\mu)=\begin{bmatrix}Q^m_k(\mu) & 0 & 0 & 0\\ 0 & R^m_k(\mu) & -T^m_k(\mu) & 0\\ 0 & -T^m_k(\mu) & R^m_k(\mu) & 0\\ 0 & 0 & 0 & Q^m_k(\mu)\end{bmatrix}, \tag{14}$$

that allows transforming (13) into the following form

$$\vec{I}_S=\frac{\Lambda}{4\pi}\oint\left(\sum_{k=0}^{\infty}(2k+1)\sum_{m=0}^{k}(2-\delta_{0,m})\left(\overleftrightarrow{\phi}_1\left(m(\varphi-\varphi')\right)\overleftrightarrow{A}\overleftrightarrow{D}_1+\overleftrightarrow{\phi}_2\left(m(\varphi-\varphi')\right)\overleftrightarrow{A}\overleftrightarrow{D}_2\right)\right)\vec{\tilde{L}}\left(\tau,\hat{\mathbf{l}}'\right)d\hat{\mathbf{l}}', \tag{15}$$

where $\overleftrightarrow{A}^{m}_{k}(\mu,\mu') = \overleftrightarrow{\Pi}^{m}_{k}(\mu)\chi_k\overleftrightarrow{\Pi}^{m}_{k}(\mu')$.

Let's consider the source function (12) from the Equation (10). It is possible to show that using (12), (15) and (14) after some tedious transformations we can get the expression for the source function

$$\vec{\Delta}(\tau,\mu,\varphi)=\sum_{m=0}^{\infty}\sum_{k=m}^{\infty}\frac{2k+1}{2}\left(\overleftrightarrow{\phi}_1(m\varphi)\overleftrightarrow{\Pi}^{k}_{m}(\mu)\overleftrightarrow{\Phi}_k(\tau)\overleftrightarrow{\Pi}^{k}_{m}(\mu_0)\overleftrightarrow{D}_1+\overleftrightarrow{\phi}_2(m\varphi)\overleftrightarrow{\Pi}^{k}_{m}(\mu)\overleftrightarrow{\Phi}_k(\tau)\overleftrightarrow{\Pi}^{k}_{m}(\mu_0)\overleftrightarrow{D}_2\right)\vec{f}_k(0), \tag{16}$$

where $\overleftrightarrow{\Phi}_k(\tau) = \frac{1}{2k+1}\left[\overleftrightarrow{\mathrm{A}}_{k+1}\overleftrightarrow{b}_{k+1}\overleftrightarrow{\Xi}_{k+1}(\tau)\overleftrightarrow{a}_k + 4\frac{(2k+1)}{k(k+1)}\overleftrightarrow{b}_k\overleftrightarrow{\Xi}_k(\tau)\overleftrightarrow{B} + \overleftrightarrow{\mathrm{A}}_k\overleftrightarrow{b}_{k-1}\overleftrightarrow{\Xi}_{k-1}(\tau)\overleftrightarrow{a}_k\right] - \overleftrightarrow{b}_k\overleftrightarrow{\Xi}_k(\tau)$, $\overleftrightarrow{b}_k = \overleftrightarrow{1} - \Lambda\overleftrightarrow{\chi}_k$, $\overleftrightarrow{\Xi}_k(\tau) = \overleftrightarrow{T}_{SC}\overleftrightarrow{Z}_k(\tau)\overleftrightarrow{T}_{CS}$, $\overleftrightarrow{a}_k = \mathrm{diag}\{k,\kappa,\kappa,k\}$, $\overleftrightarrow{\mathrm{A}}_k = \frac{\overleftrightarrow{a}_k}{k}$, $\overleftrightarrow{B} = \mathrm{diag}\{0,1,1,0\}$, $\kappa = \sqrt{k^2-4}$.

Let's present the smooth part of the solution similar to the source function (16)

$$\vec{\tilde{L}}(\tau,\mu,\varphi) = \sum_{m=0}^{\infty}\left[\overleftrightarrow{\phi}_1(m\varphi)\vec{\tilde{L}}^m_1(\tau,\mu) + \overleftrightarrow{\phi}_2(m\varphi)\vec{\tilde{L}}^m_2(\tau,\mu)\right], \tag{17}$$

that gives us two integral equations ($i = 1$, 2)

$$\mu\frac{\partial}{\partial\tau}\vec{\tilde{L}}^m_i(\tau,\mu) + \vec{\tilde{L}}^m_i(\tau,\mu) = \frac{\Lambda}{2}\sum_{k=0}^{\infty}(2k+1)\int_{-1}^{1}\overleftrightarrow{A}^{m}_{k}(\mu,\mu')\vec{\tilde{L}}^m_i(\tau,\mu')d\mu' + \vec{\Delta}_i(\tau,\mu). \tag{18}$$

with the boundary conditions

$$\vec{\tilde{L}}^m_i(0,\mu)\Big|_{\mu\geq 0} = 0,\quad \vec{\tilde{L}}^m_i(\tau_0,\mu)\Big|_{\mu\geq 0} = -\left[\vec{L}^m_a(\tau_0,\mu)\right]_i, \tag{19}$$

where $\vec{\Delta}_i(\tau,\mu) = \sum\limits_{k=0}^{\infty}\frac{2k+1}{2}\overleftrightarrow{\Pi}^{k}_{m}(\mu)\overleftrightarrow{\Phi}_k(\tau)\overleftrightarrow{\Pi}^{k}_{m}(\mu_0)\overleftrightarrow{D}_i\vec{f}_k(0)$.

4. MATRIX-OPERATOR METHOD

Since $\vec{\tilde{L}}_i^m(\tau,\mu)$ is a smooth function, the Equation (18) can be directly solved by the discrete ordinate method (DOM), but it is better to take into account the plane symmetry of the boundary value problem and to use the double Gaussian quadrature [2] for the scattering integral representation in (18)

$$\int_{-1}^{1}\vec{f}(\mu')d\mu'=\int_{-1}^{0}\vec{f}(\mu')d\mu'+\int_{0}^{1}\vec{f}(\mu')d\mu'\approx\frac{1}{2}\sum_{j=1}^{N/2}w_j\vec{f}(\mu_j^-)+\frac{1}{2}\sum_{j=1}^{N/2}w_j\vec{f}(\mu_j^+),\tag{20}$$

where $\vec{f}(\mu')=\overleftrightarrow{A}_k^m(\mu,\mu')\vec{\tilde{L}}_i^m(\tau,\mu')$, $\mu_j^+=0.5(\mu_j+1)$, $\mu_j^-=0.5(\mu_j-1)$, μ_j is the zeros of Legendre polynomial $\mathrm{P}_{N/2}(\mu_j)$=0, N is the complete number of ordinates. The upper index $\ll+\gg$ or $\ll-\gg$ by the ordinates corresponds to the cosine sign and determinates the direction, in which this stream propagates: upward or downward.

The equation set of DOM according to (20) takes the following form

$$\begin{aligned}\mu_i^\pm\frac{\partial}{\partial\tau}\vec{L}\left(\tau,\mu_i^\pm\right)&=-\vec{L}\left(\tau,\mu_i^\pm\right)+\frac{\Lambda}{4}\sum_{k=0}^{K}\sum_{j=1}^{N/2}(2k+1)w_j\left(\overleftrightarrow{A}\left(\mu_i^\pm,\mu_j^-\right)\vec{\tilde{L}}\left(\tau,\mu_j^-\right)+\overleftrightarrow{A}\left(\mu_i^\pm,\mu_j^+\right)\vec{\tilde{L}}\left(\tau,\mu_j^+\right)\right)\\&\quad+\vec{\Delta}\left(\tau,\mu_i^\pm\right),\end{aligned}\tag{21}$$

where we omitted the obvious indices c, m and k.

Now let's introduce the following designation for $4N\times4N$ matrices and $2N\times1$ vectors:

$$\overleftrightarrow{F}=\left\{\begin{array}{cc}\sum\limits_{k=0}^{K}(2k+1)\overleftrightarrow{A}(\mu_i^\pm,\mu_j^+) & \sum\limits_{k=0}^{K}(2k+1)\overleftrightarrow{A}(\mu_i^\pm,\mu_j^-)\end{array}\right\};$$

$$\overleftrightarrow{M}=\begin{bmatrix}\overleftrightarrow{\mu}^+ & \overleftrightarrow{0}\\ \overleftrightarrow{0} & \overleftrightarrow{\mu}^-\end{bmatrix};\quad \vec{L}_\pm=\begin{bmatrix}\vec{L}(\mu_1^\pm)\\ \vdots\\ \vec{L}(\mu_{N/2}^\pm)\end{bmatrix};\quad \vec{\Delta}_\pm=\begin{bmatrix}\vec{\Delta}(\mu_1^\pm)\\ \vdots\\ \vec{\Delta}(\mu_{N/2}^\pm)\end{bmatrix},$$

that allows to rewrite the equation set (21) in the matrix form

$$\frac{d}{d\tau}\begin{bmatrix}\vec{L}_+\\ \vec{L}_-\end{bmatrix}+\overleftrightarrow{B}\begin{bmatrix}\vec{L}_+\\ \vec{L}_-\end{bmatrix}=\overleftrightarrow{M}^{-1}\begin{bmatrix}\vec{\Delta}_+\\ \vec{\Delta}_-\end{bmatrix},$$

where $\overleftrightarrow{B}\equiv\overleftrightarrow{M}^{-1}(\overleftrightarrow{1}-0.25\Lambda\overleftrightarrow{F}\overleftrightarrow{W})$, $\overleftrightarrow{W}=\mathrm{diag}(w_j)$.

Taking into account the boundary conditions (19) and using the scalar transformation [8] the solution of this set can be presented as follows

$$\begin{aligned}&\begin{bmatrix}-\overleftrightarrow{u}_{12} & e^{+\overleftrightarrow{\Gamma}_-\tau_0}\overleftrightarrow{u}_{11}\\ -e^{-\overleftrightarrow{\Gamma}_+\tau_0}\overleftrightarrow{u}_{22} & \overleftrightarrow{u}_{21}\end{bmatrix}\begin{bmatrix}\vec{L}_-(0)\\ \vec{L}_+(\tau_0)\end{bmatrix}=\\&\begin{bmatrix}\vec{S}_-+e^{-\overleftrightarrow{\Gamma}_+\tau_0}\overleftrightarrow{u}_{21}\vec{L}_S\\ \vec{S}_++\overleftrightarrow{u}_{22}\vec{L}_S\end{bmatrix}+\begin{bmatrix}\overleftrightarrow{u}_{11} & -e^{+\overleftrightarrow{\Gamma}_-\tau_0}\overleftrightarrow{u}_{12}\\ e^{-\overleftrightarrow{\Gamma}_+\tau_0}\overleftrightarrow{u}_{21} & -\overleftrightarrow{u}_{22}\end{bmatrix}\begin{bmatrix}\vec{L}_+(0)\\ \vec{L}_-(\tau_0)\end{bmatrix},\end{aligned}\tag{22}$$

where $\begin{bmatrix}\vec{S}_+\\ \vec{S}_-\end{bmatrix}\equiv\begin{bmatrix}e^{\overleftrightarrow{\Gamma}_-\tau} & \overleftrightarrow{0}\\ \overleftrightarrow{0} & \overleftrightarrow{1}\end{bmatrix}\int_0^{\tau_0}\exp(\overleftrightarrow{\Gamma}t)\overleftrightarrow{U}^{-1}\overleftrightarrow{M}^{-1}\begin{bmatrix}\vec{\Delta}_+\\ \vec{\Delta}_-\end{bmatrix}dt$; $\exp(\overleftrightarrow{B}t)=\overleftrightarrow{U}\exp(\overleftrightarrow{\Gamma}t)\overleftrightarrow{U}^{-1}$, $\overleftrightarrow{\Gamma}=\begin{bmatrix}\overleftrightarrow{\Gamma}_- & \overleftrightarrow{0}\\ \overleftrightarrow{0} & \overleftrightarrow{\Gamma}_+\end{bmatrix}$, $\overleftrightarrow{\Gamma}_\pm=\mathrm{diag}\{\gamma_i\}$, $\gamma_{i-1}<\gamma_i$; $\overleftrightarrow{U}^{-1}=\begin{bmatrix}\overleftrightarrow{u}_{11} & \overleftrightarrow{u}_{12}\\ \overleftrightarrow{u}_{21} & \overleftrightarrow{u}_{22}\end{bmatrix}$. $\vec{L}_+(0)$, $\vec{L}_-(\tau_0)$ are the external radiation incidents on the slab from above and below correspondingly.

The solution of Equation (22) can be presented in the following form

$$\begin{bmatrix}\vec{L}_-(0)\\ \vec{L}_+(\tau_0)\end{bmatrix}=\begin{bmatrix}\vec{J}_-\\ \vec{J}_+\end{bmatrix}+\begin{bmatrix}\overleftrightarrow{R}_- & \overleftrightarrow{T}_-\\ \overleftrightarrow{T}_+ & \overleftrightarrow{R}_+\end{bmatrix}\begin{bmatrix}\vec{L}_+(0)\\ \vec{L}_-(\tau_0)\end{bmatrix},\tag{23}$$

where $\begin{bmatrix}\vec{J}_- \\ \vec{J}_+\end{bmatrix} = \overleftrightarrow{H}\begin{bmatrix}\vec{S}_- + e^{-\overleftrightarrow{\mathbf{\Gamma}}_+\tau_0}\overleftrightarrow{u}_{21}\vec{L}_S \\ \vec{S}_+ + \overleftrightarrow{u}_{22}\vec{L}_S\end{bmatrix}$, $\begin{bmatrix}\overleftrightarrow{R}_- & \overleftrightarrow{T}_- \\ \overleftrightarrow{T}_+ & \overleftrightarrow{R}_+\end{bmatrix} = \overleftrightarrow{H}\begin{bmatrix}\overleftrightarrow{u}_{11} & -e^{+\overleftrightarrow{\mathbf{\Gamma}}_-\tau_0}\overleftrightarrow{u}_{12} \\ e^{-\overleftrightarrow{\mathbf{\Gamma}}_+\tau_0}\overleftrightarrow{u}_{21} & -\overleftrightarrow{u}_{22}\end{bmatrix}$,

$\overleftrightarrow{H} = \begin{bmatrix}-\overleftrightarrow{u}_{12} & e^{+\overleftrightarrow{\mathbf{\Gamma}}_-\tau_0}\overleftrightarrow{u}_{11} \\ -e^{-\overleftrightarrow{\mathbf{\Gamma}}_+\tau_0}\overleftrightarrow{u}_{22} & \overleftrightarrow{u}_{21}\end{bmatrix}^{-1}$.

In the case of the semi-infinite slab $\tau_0 \to \infty$: $\vec{J}_+ = 0$, $\overleftrightarrow{T}_- = \overleftrightarrow{T}_+ = 0$, $\vec{L}_-(\tau_0) = \vec{L}_+(\tau_0) = 0$, and it is possible to write down the solution of (23) at once

$$\vec{L}_-(0) = -\overleftrightarrow{u}_{12}^{-1}\vec{S}_-. \tag{24}$$

The expression (24) is the solution of Milne-Ambartsumian problem for the arbitrary scattering phase function taking into account the polarization effect.

Using the matrix-operator method and the Equation (23) one can transfer to VRTE solution for the vertically stratified turbid medium slab. In this case it is possible to replace two adjacent slabs by one with some effective parameters, where the radiation transfer will be described by the expression like (23) — the principle of invariance. This approach allows easily including into the consideration the radiation transfer through the interface between two slabs with refraction [2, 9].

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